Partition Properties for Non-Ordinal Sets Under the Axiom of Determinacy

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The Simplest Combinatorics

- The Pigeonhole Principle: If $m < n \in \mathbb{N}$ and $f : n \to m$ is a partition of n into m-pieces, then for some $i < m$, $f^{-1}(i)$ is bigger than 1. (Dirichlet 1834, "Schubfachprinzip")
- Ramsey's theorem: Fix $m, k, l \in \mathbb{N}$. Then there is an $n \in \mathbb{N}$ so that whenever $f : [n]^k \to m$ is a partition of the increasing k-tuples from n into m-pieces, then there is an $A \subseteq n$ so that A has size l and f is constant on $[A]^k$. (Ramsey, 1930)

The Coloring Picture

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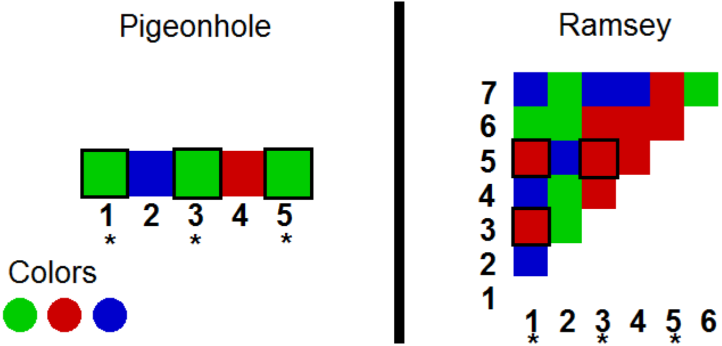
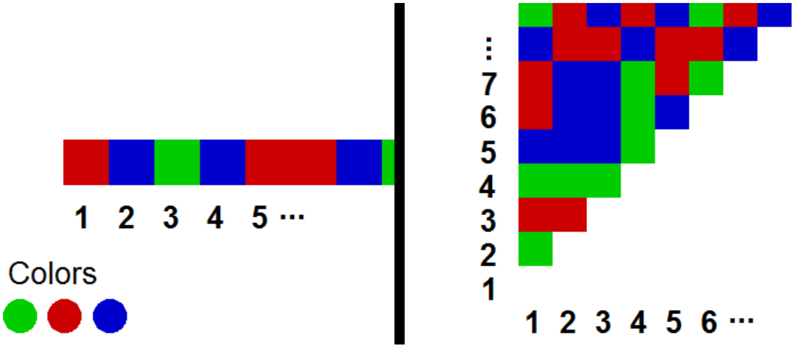
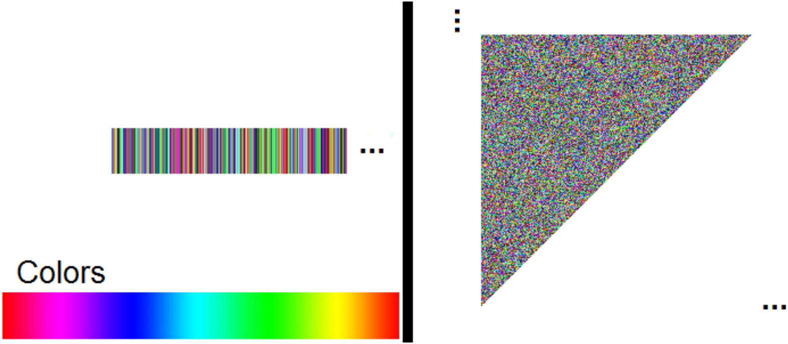


A Bigger Canvas



A More Diverse Palette



More Complicated Combinatorics

Definition

For any set A, $[A]^n = \{s \subseteq A : |s| = n\}$ and $[A]^{<\omega} = \bigcup_{n\in\omega}[A]^n$.

Definition

Let A and B be infinite sets.

- (A, B) has the **Ramsey** property iff for any $f : [A]^{<\omega} \to B$, there is an $X \subseteq A$ so that $|X| = |A|$ and f is constant on each $[X]^n$.
- (A, B) has the **Rowbottom** property iff for any $f : [A]^{<\omega} \to B$, there is an $X \subseteq A$ so that $|X| = |A|$ and $f[[X]^{<\omega}]$ is countable.
- (A, B) has the **strong Jónsson** property iff for any $f : [A]^{<\omega} \to B$, there is an $X \subseteq A$ so that $|X| = |A|$ and

$$|B - f[[X]^{<\omega}]| = |B|.$$

Obstructions Under the Axiom of Choice

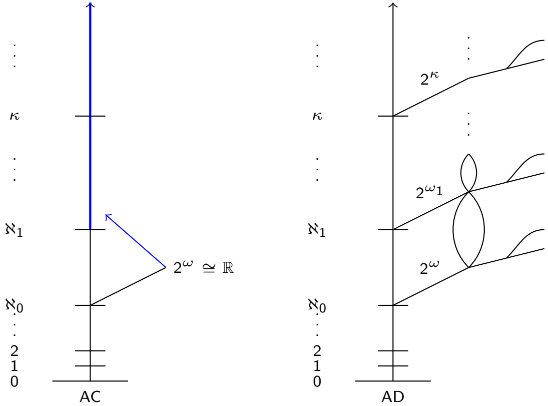
If all possible colorings are considered, including those only constructable with the axiom of choice, then the existence of a non-trivial pair with any of these three properties is outside of the scope of classical mathematics (They are equiconsistent and between the existence of a measurable cardinal and $0^{\#}$).

The colorings responsible for denying these properties are kind of like non-measurable sets. To further explore the question of the existence of these pairs, we can restrict our attention to definable functions.

Formally, we take definable to mean the coloring is a function in $L(\mathbb{R})$, where the axiom of determinacy (AD) is true.

Definable Functions and Size

A consequence of only using definable functions and measuring the size of sets with injections is that the cardinality structure is fundamentally altered.



The Original Inspiration

Recall that Θ is the least cardinal that $\mathbb{R}$ does not surject onto.

In 2015, S. Jackson, R. Ketchersid, F. Schlutzenberg, and W.H. Woodin proved the following:

Theorem

Assume AD and $V = L(\mathbb{R})$. Let $\lambda < \kappa < \Theta$ be uncountable cardinals. Then

1. *If $cf(\kappa) = \omega$ or κ is regular, then (κ, λ) has the Rowbottom property.*
2. *(κ, λ) has the strong Jónsson property.*

Additionally, it is an easy corollary of work of J. Steel that in $L(\mathbb{R})$, if $\kappa < \Theta$ is a regular cardinal, then $(\kappa, 2)$ has the Ramsey property.

Non-Ordinal Infinite Sets

In the definable context, the most obvious example is $\mathbb{R}$. Quotients, unions, and products can be used to produce other examples. The examples we understand best are sets formed from finite unions and products of uncountable cardinals (below Θ), $\mathbb{R}$, and ${}^{\mathbb{R}}/_{\mathbb{Q}}$. Denote the collection of all sets constructed in this manner by $\mathcal{X}$.

My results for these are as follows (Assuming AD and $V = L(\mathbb{R})$):

- (A, B) has the strong Jónsson property for all $A, B \in \mathcal{X}$,
- $({}^{\mathbb{R}}/_{\mathbb{Q}}, \mathbb{R})$ has the Ramsey property, and
- if κ is a cardinal, then $(\mathbb{R}, \kappa)$ has the Rowbottom property and $({}^{\mathbb{R}}/_{\mathbb{Q}}, \kappa)$ has the Ramsey property.

Future Work: Infinite Unions

The following is a preliminary report:

- if $A \in \mathcal{X}$ and B is a well-ordered unions of smooth quotients of $\mathbb{R}$, then (A, B) is Jónsson, and
- if A is a well-ordered unions of smooth quotients of $\mathbb{R}$, then there is an α so that $2^\omega \hookrightarrow A \hookrightarrow 2^\alpha$. Even with ω_1-length unions, A could be $\omega_1 \cup \mathbb{R}$, $\omega_1 \times \mathbb{R}$, $\mathbb{R}$, or maybe something else altogether.

Thanks For Listening!